The Geodesic Dynamic Relaxation Method for Problems of Equilibrium with Equality Constraint Conditions

Masaaki Miki\textsuperscript{1,3}\textsuperscript{*}, Sigrid Adriaenssens\textsuperscript{2}\textsuperscript{*}, Takeo Igarashi\textsuperscript{3,4}\textsuperscript{*}, Ken’ichi Kawaguchi\textsuperscript{5}

\textsuperscript{1}Department of Architecture, The University of Tokyo, Japan
\textsuperscript{2}Department of Civil and Environmental Engineering, Princeton University
\textsuperscript{3}Japan Science and Technology Agency/ERATO
\textsuperscript{4}Department of Computer Science, The University of Tokyo, Japan
\textsuperscript{5}Institute of Industrial Science, The University of Tokyo, Japan

SUMMARY

This paper presents an extension to the existing Dynamic Relaxation method to include equality constraint conditions in the process. The existing Dynamic Relaxation method is presented as a general, gradient-based, minimization technique. This representation allows for the introduction of the projected gradient, discrete parallel transportation and pull back operators that enable the formulation of the Geodesic Dynamic Relaxation method, a method which accounts for equality constraint conditions. The characteristics of both the existing and the Geodesic Dynamic Relaxation methods are discussed in terms of the system’s conservation of energy, damping (viscous, kinetic and drift) and geometry generation. Particular attention is drawn to the introduction of a novel damping approach named drift damping. This technique is essentially a combination of viscous and kinetic damping. It allows for a smooth and fast convergence rate in both the existing and the Geodesic Dynamic Relaxation processes. The case study was performed on the form-finding of an iconic, ridge-and-valley, pre-stressed membrane system, which is supported by masts. The study shows the potential of the proposed method to account for specified (total) length requirements. The Geodesic Dynamic Relaxation technique is widely applicable to the form-finding of force-modelled systems (including mechanically and pressurized pre-stressed membranes) where equality constraint control is desired. Copyright © 0000 John Wiley & Sons, Ltd

Summary continued...
have included alternative membrane (Gosling and Lewis [14], Hegyi et al. [15]) and pneumatic (Rodriguez et al. [16]) elements, non-regular tensegrity modules (Zhang et al. [17], Bel Hadj Ali et al. [18]), reciprocal frames links (Douthe and Baverel [19]), pulley elements with friction (Hincz [20]) and beam and torsion elements (Adriaenssens and Barnes [20, 21]).

The method is based on Newton’s second law of motion and follows the movement of each node of a structure, for small time intervals, from its initial position until all vibrations have become negligible due to artificial damping. Although the name DR contains the term dynamic, it is most widely used as a computational method to solve static problems. Due to the law of inertia, it solves equilibrium problems more efficiently than the steepest descent method, which is a standard, gradient-based, minimization method. Similar to the steepest descent method, the DR method also evaluates the gradient of energy function only. Therefore, it has potential to be a powerful alternative of steepest descent method.

Little research has been performed on the inclusion of equality conditions constraints into this DR process. Such constraints are mostly attributed to physical and geometric limitations of the technical nature of a project. To include these constraints into the DR method, developments have included formulations for uniform net meshes [22], element distortion control [11], nodal planarity [23] and length prescription [24].

The equality condition constraints discussed in this paper are different from those based on geometric construction limitations; they provide an alternative way to model and control key structural elements in a force-modelled system. For example, compression struts in a pre-stressed membrane roof structure could be modelled as elements with a specified elastic stiffness. Alternatively, it might be reasonable to model these struts with length constraints while the other components (such as the pre-stressed cables or membranes) are treated as components with specified elastic stiffnesses. Similarly, it might be beneficial to model specific pre-stressed cables with total length constraints. Another clear example for desirable inclusion of equality constraint conditions in the DR process relates to the modeling of air-supported, pneumatic structures, in which the air is treated as a volume constraint. From an engineering viewpoint, the axial forces, in length-controlled struts or cables, and the pressure acting on the membrane, in a pneumatic with a constant air volume, should be taken into consideration in the DR process. These forces can be considered reaction forces produced by the equality constraint conditions.

The arising research question becomes: how can equality constraint conditions be incorporated in the DR process while appropriately accounting for the reaction forces produced by these conditions?

The remainder of the paper is organized as follows. Section 2 describes the existing DR process as a general, gradient-based, minimization technique and discusses its characteristics. In section 3, the Geodesic Dynamic Relaxation method is presented. This technique allows for the introduction of equality constraint conditions. Its features are discussed in section 4. Section 5 shows the accuracy and validity of the presented developments based on the case study of an existing, pre-stressed, membrane system. In section 6, conclusions and a summary of the paper are given.

2. DYNAMIC RELAXATION PROCESS

The DR method could be considered as a general, gradient-based, minimization approach that solves

\[
\text{find } x \in \mathbb{R}^n \mid f(x) \rightarrow \min, \tag{1}
\]

where \( x \) represents unknown variables, \( f \) is a real-valued function of \( x \), \( \mathbb{R}^n \) is the set of \( n \)-dimensional real vectors, and \( n \) denotes the total number of unknown variables. In problems of equilibrium, \( f \) might represent the sum of elastic energies plus additional potential functions. In this paper, we use an \( n \)-dimensional column vector to represent \( x \), such as

\[
x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T.
\]

We only consider problems of equilibrium. As each local minimum represents an equilibrium state of the system, we study local, not global, minima of the function \( f \). A local minimum is a point \( x \in \mathbb{R}^n \) that satisfies the stationary condition of the function or

\[
\nabla f = 0, \tag{2}
\]
where $\nabla f$ is the gradient of $f$ defined as

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

and $0$ is an $n$-dimensional row vector of which all the components are set to $0$.

In this paper we use an $n$-dimensional row vector to represent gradient vectors. In order to emphasize the fact that $\nabla f$ is a function of $x$, we write

$$\omega(x) = \nabla f.$$  

Note that, in problems of equilibrium, $\omega(x)$ represents external force acting on a single particle whose position is denoted by $x$. In typical problems of equilibrium that are solved by DR, $x$ is the set of $x$, $y$ and $z$ coordinates of all the free nodes. In those cases, $\omega(x)$ can be decomposed to nodal forces which act on individual nodes. As a result, the problem of equilibrium discussed in this paper takes the form

$$\text{find } x \in \mathbb{R}^n \mid \omega(x) = 0.$$  

To further develop these expressions for DR, we initially consider the Steepest Descent Method (SDM), which is the first standard, gradient-based, minimization approach. For a given initial configuration $x_0$, SDM iteratively generates a point series $\{x_0, \ldots, x_t, \ldots\}$ based on

$$r_t = -\omega(x_t)^T,$$

$$x_{t+1} = x_t + \alpha r_t,$$  

(6)

where $\alpha$ is a step-size factor, $t$ is the step number, and $r_t$ is the standard search direction. By keeping the step-size $\alpha$ constant, sufficient convergence efficiency cannot be achieved. Therefore, $\alpha$ is often determined by a line search algorithm (e.g. see Fletcher [25], section 2.6). However, a typical line search algorithm calls for more than one iteration in each line search. As a result, not only $\omega(x)$, but also $f(x)$, are typically evaluated and taken into account. Because the original SDM, without line search, only assesses $\omega(x)$, it would be preferable to consider the integration of SDM with a typical line search algorithm, which only evaluates $\omega(x)$ and not $f(x)$.

The DR method can be introduced as a technique for this natural integration. In DR, instead of performing a line search, the point series $\{x_0, \ldots, x_t, \ldots\}$ tends to move along a straight line by introducing a velocity parameter, denoted by $q$, and represented in an $n$-dimensional column vector.

For a given initial configuration, $\{x_0, q_0\}$, DR generates a point and vector series $\{\{x_0, q_0\}, \ldots, \{x_t, q_t\}, \ldots\}$ based on

$$r_t = -\omega(x_t)^T,$$

$$q_{t+1} = \gamma q_t + \beta r_t,$$

$$x_{t+1} = x_t + \beta q_{t+1},$$  

(7)

where $\beta$ is a step-size factor, and $\gamma$ is a damping coefficient. The DR process can be intuitively understood as a discrete Newton’s equation of motion by considering $\omega$, $r_t$, $q_t$, and $x_t$ as the force, acceleration, velocity, and position of a particle moving in an $n$-dimensional Euclidean space. It should be noted that the name of this method differs between communities to community and has been named Verlet’s scheme [26] and Symplectic Euler method [27–31].

2.1. Conservation of energy

Empirically, the energy conservation law is observed in DR. We demonstrate this phenomenon with the function

$$f(x, y) = (10 + r (\cos (8 \theta + 30r)))^2,$$  

(8)

where $(r, \theta)$ is a polar coordinate which can be converted from $(x, y)$ as follows:

$$r = \sqrt{x^2 + y^2},$$

$$\theta = \begin{cases} \pi & (x = 0) \\ \atan(y/x) & \text{otherwise}. \end{cases}$$  

(9)
To exclude the possibility that the observed energy conservation law occurs as a coincidence, we created this function to be sufficiently complex. Following the general manner described above, \( x \) and \( y \) are packed together as \( x = [ x \ y ]^T \).

Starting with \( x_0 = [ 1 \ 0 ]^T \) and \( q_0 = [ 0 \ 0 ]^T \), we minimize Equation (8) using DR and generate a point series \( \{ x_0, \ldots, x_t, \ldots \} \). We use \( \beta = 0.005 \) for the step-size. By setting the damping coefficient \( \gamma \) to 1.0, the point series does not converge nor diverge as shown in Figure 1 (a). This observation suggests that certain quantities are conserved. We define the kinetic energy, \( K_t \), and the total energy, \( E_t \), at step \( t \) by

\[
K_t = \frac{1}{2} q_t^T q_t, \quad E_t = K_t + f(x_t). \tag{10}
\]

As shown in Figure 1 (b), the value \( E_t \) is roughly kept constant throughout the computational process, and it does not seem flat in a strict sense. However, tracing the trajectories of \( f \) and \( K \), in the same plot, shows a path with larger magnitudes which cancel each other out. Similar observations were reported and explained in studies on the Symplectic Euler method [27–31]. When \( \gamma = 0.0 \), the DR becomes an exact SDM, in which the step size of SDM is given by \( \alpha = \beta^2 \). The DR obeys the law of conservation of energy, when \( \gamma = 1.0 \).

![Figure 1. The energy conservation law observed in the DR process: (a) plot of the generated point series superimposed on a contour of the function and (b) plot of time steps versus total energy \( E \) and kinetic energy \( K \).](image)

### 2.2. Damping coefficient

The basis of the DR method is to follow the movement of each node of the structure, for small time intervals, from its initial position until all vibrations have died out due to artificial damping.

Usually, a finite element form of DR is described as being the means by which any unstable discretized system might be brought to rest through the application of viscous damping, applied through the damping coefficient \( \gamma \), of the nodal movements. In order to achieve the most rapid convergence, the lowest mode of vibration of the structure is critically damped and the fictitious nodal mass components are adjusted to be proportional to the corresponding direct stiffness components. In some cases, the critical damping coefficient, \( \gamma \), might be difficult to estimate. Starting from a rather inaccurate initial position, certain elastic members are grossly deformed in the initial DR stages and induce locally unbalanced forces and related high frequency modes [3]. Therefore, additional control measures such as varying mass components, locally varying damping constants and fictitious member stiffnesses are needed in different stages of the analysis to obtain convergence [32]. Frieze, Hobbs and Dowling [33] used similar variable controls for
the investigation of plate buckling; significantly different levels of damping were needed for varying levels of applied loading. Zhang and Yu [34] have presented a modified adaptive Dynamic Relaxation Methods based on the viscous damping approach. In this case, the damping coefficient is based on a function of the current system configuration, the internal element force and the mass matrix. Recently, Rezaee-pajand et al. [35] proposed a method that minimizes errors between two successive iterations to obtain optimum fictitious mass and viscous damping with the aid of the Stodola iterative process.

While working on unstable, geo-mechanical problems, Cundall [36] first suggested using kinetic damping, which proved to be entirely stable and rapidly converging when dealing with large unbalanced forces [3]. Since viscous damping is neglected, there is no need for prior determination of the damping constant. The underlying basis of kinetic damping is that as an oscillating body passes through a minimum potential energy state, its total kinetic energy $K_t$ reaches a local maximum. Upon detection of this local peak, all current nodal velocities are set to zero. The process is then restarted from the current geometry and continued through generally decreasing peaks until all energy of all modes of vibration has been dissipated and the structure reaches its static equilibrium state. For a comprehensive overview of kinetic damping in DR, the reader is referred to Shugar [37].

In this section, we present a framework that expresses both viscous and kinetic damping as similar parameters. First, we define $\theta_t$, an entity between the acceleration and the velocity at step $t$ by

$$\theta_t = \frac{q_t^T r_t}{|q_t||r_t|}$$

(11)

where $|\cdot|$ is the standard Euclidean norm. Note that $\theta_t = 1.0$ when the acceleration and the velocity point in the same direction, $\theta_t = 0.0$ when one is perpendicular to the other, and $\theta_t = -1.0$ when they point in opposite directions. Second, we assume that the damping coefficient at each step is given by a function of $\theta_t$, i.e.,

$$\gamma_t = \gamma(\theta_t)$$

(12)

This function gives a characteristic curve between $\theta$ and $\gamma$, and determines the behavior of DR. By optimizing this function, the performance of the DR methods can, thus, further be improved.

In this context, viscous and kinetic damping can be described as follows. Viscous damping is characterized by

$$\gamma(\theta) = \text{const.}$$

(13)

The trajectory and the history of the energy in DR with viscous damping ($\gamma = 0.98$) applied to Equation (8) is shown in Figure 2.

On the other hand, kinetic damping is characterized by

$$\gamma(\theta) = \begin{cases} 
1.0 & (1 \geq \theta > 0) \\
0.0 & (0 \geq \theta \geq -1)
\end{cases}$$

(14)

These equations can be explained as follows. When kinetic energy $K_t$ reaches a local maximum, the acceleration $r_t$ becomes orthogonal to the velocity $q_t$, i.e., $\theta = 0$. Prior to that instance, $\theta$ is greater than 0 because the kinetic energy increases. Hence, setting $\gamma$ to 0, when $\theta \leq 0$, describes kinetic damping. In other words, the system accelerates with no damping until acceleration and velocity occur perpendicular to each other. Once they are perpendicular to each other, the system decelerates instantly.

The trajectory and the history of energies in the DR process, with kinetic damping applied to Equation (8), are shown in Figure 3. Although the approach with kinetic damping proves to be the most advantageous in the majority of numerical examples, this example shows that adopting kinetic damping in the DR process might bring about an inefficient trajectory due to the discontinuity in the characteristic curve.

In addition to viscous and kinetic damping, we introduce the concept of drift damping, which is defined by

$$\gamma(\theta) = 0.95 + \theta/20$$

(15)
This function is developed to solve the discontinuity in kinetic damping. The trajectory and the history of the energies in the DR process, with drift damping applied to Equation (8) are shown in Figure 4. By using drift damping, the phase of DR process is dynamically adjusted between acceleration and deceleration as well as kinetic damping. Because Equation (15) adjusts $\gamma$ from 1.0 to 0.90 smoothly, as opposed to kinetic damping, the phase transition in drift damping is gentle rather than instant. Note that, because the effect of damping works exponentially, the damping effect with $\gamma = 0.92, 0.95$ and 0.98 works in very different ways. Table I shows that, after 100 step iterations, a pulse force at the first step almost vanishes if the damping coefficient is 0.92 or 0.95. However, 10% of that pulse force is still present after 100 iterations, when the damping coefficient has a value of 0.98.

Based on trial runs we pose that a desirable duration of pulse force is around 100 iterations. The bias degree, 1/20, in Equation (15) is adjusted to achieve this duration. In sections 4 and 5, $\gamma = 0.98$ is used for viscous damping. This particular number 0.98 is also adjusted to ensure that the initial pulse force is dissipated after 100 iterations. Therefore, if a different expected duration of pulse force is chosen, Equation (15) or the constant value in viscous damping may be changed.

In Figure 5 the three $\gamma - \theta$ relation curves for kinetic, viscous and drift damping are plotted. The flowchart for the DR method combined with the proposed flexible framework of damping is shown in Figure 6. The DR process changes its performance according to different parameter choices in Equations (13) and (15). A detailed explanation of their respective performances is provided in Appendix A for a benchmark test. All three damping approaches are applied to the geodesic DR framework presented in section 3.

![Figure 2](image.png)

Figure 2. Viscous Damping ($\gamma = 0.98$): (a) plot of the generated point series superimposed on a contour of the function and (b) plot of time steps versus total energy $E$ and kinetic energy $K$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.92</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{100}$</td>
<td>2.39E-04</td>
<td>0.592E-03</td>
<td>1.32E-01</td>
</tr>
<tr>
<td>$t$ that gives $\gamma^t = 0.1$</td>
<td>27</td>
<td>44</td>
<td>113</td>
</tr>
</tbody>
</table>

Table I. For three different damping coefficients $\gamma$, the strength of a pulse force present after 100 iterations $\gamma_{100}$ and number of iterations $t$ at which the pulse force is reduced to 10% of its initial value are given.

2.3. Straight line geometry generation

By setting $r$ and $\gamma$ to 0 and 1.0 respectively in DR, i.e.,

$$r_t = 0$$
$$q_{t+1} = q_t + \beta r_t$$
$$x_{t+1} = x_t + \beta q_{t+1}$$

(16)

Total Energy $E = f + K$

Kinetic energy $K$
an important feature of the DR process is revealed. Starting from $q_0$, which does not equal 0, and an arbitrary $x_0$, the DR process generates a straight line on search space $\mathbb{R}^n$, i.e., $x_t = x_0 + \beta t q_0$. In terms of Newtonian mechanics, this is called the law of inertia. Note that this straight line does not correspond to a line drawn on a physical structure such as a membrane roof, but a point series defined on $\mathbb{R}^n$. Hence, before the force is applied, the DR method keeps generating a straight line.
When a force $\omega(x)$ is applied to the system, such a straight line is gradually altered based on $\omega(x)$ step by step.

This preliminary discussion of the DR process (section 2) highlights particular characteristics that are worth discussing when handling equality constraint conditions, which are addressed in the section 3. These features can be summarized as (i) the DR algorithm only evaluates $\omega(x)$, and not $f(x)$, (ii) with no damping applied, the total energy of the force-modelled system is conserved, (iii) viscous, kinetic and drift damping can successfully be applied to the DR process and (iv) the DR process simply generates a straight line when no force or strain energy is applied to the system.

3. GEODESIC DYNAMIC RELAXATION METHOD

In this paper, we present an extension to the DR method to allow for equality constraint conditions to be incorporated into the process. Problems of equilibrium with equality constraint conditions are typically defined as

$$
\text{find } x \in \mathbb{R}^n \mid f(x) \rightarrow \min,
\begin{cases}
g_1(x) = \bar{g}_1 \\
\vdots \\
g_m(x) = \bar{g}_m
\end{cases}
$$

where $m$ is the total number of constraint conditions, $g_1(x), \ldots, g_m(x)$ are quantities to be constrained, and $\bar{g}_1, \ldots, \bar{g}_m$ are the values to which the quantities are constrained. In this paper, we assume $m < n$ and that $g_1(x), \ldots, g_m(x)$ have gradients $\nabla g_1, \ldots, \nabla g_m$.

As we consider problems of equilibrium, we only have to consider a local minimum which satisfies the stationary condition of a Lagrangian composed of this problem. The Lagrangian of this problem is given by

$$
\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j (g_j(x) - \bar{g}_j),
$$
where $\lambda = [\lambda_1, \cdots, \lambda_m]$ represents the Lagrange multipliers. Note that we use an $m$-dimensional row vector to represent the Lagrange multipliers. The stationary condition of $\mathcal{L}(x, \lambda)$ is given by

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \text{ and } \frac{\partial \mathcal{L}}{\partial \lambda} = 0.$$  
(19)

For functions of both $x$ and $\lambda$, we define $\nabla f(x, \lambda) = \frac{\partial f}{\partial x}$ and exclude the partial derivatives with respect to the Lagrange multipliers from $\nabla f$. Then, the first and second stationary conditions are respectively expanded as

$$\nabla \mathcal{L} = \nabla f + \sum_{j=1}^{m} \lambda_j \nabla g_j = 0 \text{ and } \begin{cases} g_1(x) = \bar{g}_1 \\ \vdots \\ g_m(x) = \bar{g}_m \end{cases}.$$  
(20)

The first condition represents an equilibrium state between $\nabla f$ and the reaction forces supplied by the constrained quantities. The second condition is the given set of equality constraint conditions. For example, if $g_j(x)$ represents a length of a strut, volume of air, or angle of a hinge joint, $\lambda_j$ represents axial force, pressure or moment acting on those structural components, respectively.

We can regard $\mathbb{R}^n$ as an $n$-dimensional, Euclidean space by considering the elements $x \in \mathbb{R}^n$ as points of which coordinates are $(x_1, \cdots, x_n)$. By collecting all the points in $\mathbb{R}^n$ that satisfy all the constraint conditions, we define

$$\bar{S} \equiv \{ X \in \mathbb{R}^n \mid g_j(X) = \bar{g}_j, \forall j = 1, \cdots, m \},$$  
(21)

which forms an isomanifold in $\mathbb{R}^n$. We call the $\bar{S}$ constraint isomanifold. In order to emphasize the fact that $\nabla \mathcal{L}$ is a function of both $x$ and $\lambda$, we write

$$\tau(x, \lambda) = \nabla \mathcal{L} = \omega(x) + \lambda \mathbf{J}(x),$$  
(22)

where

$$\omega(x) = \nabla f \text{ and } \mathbf{J}(x) = \begin{bmatrix} \nabla g_1 \\ \vdots \\ \nabla g_m \end{bmatrix}.$$  
(23)

In problems of equilibrium with equality constraint conditions, three different forces are taken into account: $\omega(x)$ as an external, $\lambda \mathbf{J}(x)$ as reaction, and $\tau(x, \lambda)$ as resultant force. Note that $\mathbf{J}$ is an $m \times n$ matrix and is often called a Jacobian matrix. In this paper, we assume that $\mathbf{J}$ is a full-rank matrix. Hence, as we assumed that $m < n$, the rank of $\mathbf{J}$ should be $m$.

Consequently, the problems of equilibrium with equality constraint conditions typically take the form

$$\text{find } \{ x \in \bar{S} \subset \mathbb{R}^n, \lambda \in \mathbb{R}^m \} \mid \tau(x, \lambda) = 0.$$  
(24)

Unlike $\mathbb{R}^n$, the constraint isomanifold $\bar{S}$ is assumed to be a curved subspace of $\mathbb{R}^n$. On such a curved subspace, straight lines do not exist, but geodesics can be used as an alternative to straight lines. Therefore, this extension to the existing DR method is named the Geodesic Dynamic Relaxation method.

In the geodesic DR, each iteration is extended to

$$r_t = -\phi(\omega(x_t))^T$$

$$q_{t+1} = \gamma \varphi(q_t) + \beta r_t,$$

$$x_{t+1} = \psi(x_t + \beta q_{t+1}),$$

where $\phi(\omega), \varphi(q)$ and $\psi(x)$ are projection operators to project force, velocity and position vectors to appropriate subspaces. More specifically, these terms are called projected gradient, discrete parallel transportation, and pull back, respectively.
The relationship between the geodesic and existing DR method is very similar to the one between the projected gradient method [25, 38, 39] and SDM. The projected gradient method iteratively generates a point series \( \{ x_0, \ldots, x_t, \ldots \} \) based on

\[
    r_t = -\phi ( \omega ( x_t ) )^T, \quad x_{t+1} = \psi ( x_t + \alpha r_t )
\]

where \( \phi ( \omega ) \) and \( \psi ( x ) \) are exactly the same operators as the projected gradient and pull back operators in the geodesic DR iteration. In addition, the similarity between the function of pull back operator and the fast projection method [40] is noted.

Therefore, the key contribution in this paper is to present the discrete parallel transportation \( \phi ( q ) \). This guarantees the conservation of the total energy of the system during the DR process when the damping coefficient \( \gamma \) is set to 1.0.

Before describing these three projection operators, we introduce a pseudo inverse matrix of \( J \) [41], which will be denoted by \( J^+ \), and clarify the geometry underlying the problem defined in Equation (17).

### 3.1. Underlying geometry and pseudo inverse matrix

By varying \( g_1, \ldots, g_m \) arbitrarily, the isomannifolds \( \tilde{S} \) can be moved in \( \mathbb{R}^n \), and all possible isomannifolds can be collected. Once empty manifolds are deleted, the remaining isomannifolds cover whole \( \mathbb{R}^n \) with no overlap. Hence, any point \( x \in \mathbb{R}^n \), belongs to one of those isomannifolds. Such an isomannifold specified by a given \( x \in \mathbb{R}^n \) is identified by

\[
    S ( x ) \equiv \{ X \in \mathbb{R}^n \mid g_j ( X ) = g_j ( x ) , \quad \forall j = 1, \ldots, m \}.
\]

This is a set of all the points that give the same values of \( g_1 ( x ), \ldots, g_m ( x ) \).

We use a pseudo inverse of \( J \) in order to represent the tangent space and its orthogonal complement of \( S ( x ) \) at \( x \). Although the pseudo inverse matrix is defined in a more general way, in this work, we compute the pseudo inverse matrix of \( J \) by

\[
    J^+ = J^T ( J J^T )^{-1}.
\]

This operation is allowed only when \( m < n \) and \( J \) is a full-rank matrix. It is likely that the geodesic DR can also be performed in more general cases, in which \( J^+ \) is computed in a more general way. However, the discussion of more general cases is outside the scope of this paper. In the presented case, it is obvious that \( J J^T = I_m \), where \( I_m \) is an \( m \times m \) unit matrix. However, contrary to expectations, \( J^+ J \neq I_n \), where \( I_m \) is an \( n \times n \) unit matrix. This means that \( J^+ \) is a right inverse and not a perfect inverse.

On \( S ( x ) \), we define the following four vector spaces at \( x \):

\[
    T S ( x ) = \{ \alpha \in \mathbb{R}^n_{\text{col}} \mid \alpha = ( I_n - J^+ J ) \beta , \quad \exists \beta \in \mathbb{R}^n_{\text{col}} \},
\]

\[
    T^* S ( x ) = \{ \alpha \in \mathbb{R}^n_{\text{row}} \mid \alpha = \beta ( I_n - J^+ J ) , \quad \exists \beta \in \mathbb{R}^n_{\text{row}} \},
\]

\[
    O S ( x ) = \{ \alpha \in \mathbb{R}^n_{\text{col}} \mid \alpha = J^+ J \beta , \quad \exists \beta \in \mathbb{R}^n_{\text{col}} \},
\]

\[
    O^* S ( x ) = \{ \alpha \in \mathbb{R}^n_{\text{row}} \mid \alpha = \beta J^+ J , \quad \exists \beta \in \mathbb{R}^n_{\text{row}} \},
\]

where \( \mathbb{R}^n_{\text{col}} \) and \( \mathbb{R}^n_{\text{row}} \) are the sets of \( n \)-dimensional row and column vectors respectively. From the viewpoint of differential geometry, both \( T S ( x ) \) and \( T^* S ( x ) \) represent a tangent space of \( S ( x ) \) at \( x \). If \( T^* S ( x ) \) is differentiated from \( T S ( x ) \), it would be called a cotangent space. Additionally, \( O S ( x ) \) and \( O^* S ( x ) \) are the orthogonal complements of them. Note that \( T S ( x ) \oplus O S ( x ) = \mathbb{R}^n_{\text{col}} \) and \( T^* S ( x ) \oplus O^* S ( x ) = \mathbb{R}^n_{\text{row}} \) because \( ( I_n - J^+ J ) + J^+ J = I_n \). The tangent spaces are the sets of directions that do not change the values of \( g_1 ( x ), \ldots, g_m ( x ) \). The orthogonal complements are the sets of directions that change the values of \( g_1 ( x ), \ldots, g_m ( x ) \) most effectively.
Adopting the same approach, \( T\tilde{S}, T^*\tilde{S}, O\tilde{S}, \) and \( O^*\tilde{S} \) are also defined on \( \tilde{S} \). However, we should not assume that \( x_i \) belongs to \( \tilde{S} \) at each step. Instead, we define the three projection operators on any \( S(x) \) and later employ an iterative strategy (i.e., pull-back) to superimpose \( S(x) \) onto \( \tilde{S} \).

3.2. Projected gradient operator

As shown in Figure 7, the projected gradient operator \( \phi(\omega) \) projects external force \( \omega \) to \( T^*S(x) \). This projected gradient is defined by

\[
\phi(\omega) = \omega \left( I_n - J^+J \right).
\]

As the result, \( \phi(\omega) \) always points in the direction that does not change the values of \( g_1(x), \ldots, g_m(x) \). In other words, the orthogonal component of external force to the tangent space is necessarily eliminated. As the result, acceleration \( r_i \) in the geodesic DR method also points in the direction that does not change the values of \( g_1(x), \ldots, g_m(x) \).

Incidentally, the projected gradient method can be understood as a composite function of \( \tau(x, \lambda) \) with a multiplier estimate given by

\[
\lambda(x) = (-\omega J^+) \tag{34}
\]

By directly substituting Equation (34) into Equation (22), the projected gradient \( \phi(\omega) \) is obtained. As shown by Figure 7, the projected gradient \( \phi(\omega) \) can be decomposed as two forces, \( \omega \) and orthogonal reaction force \( \lambda J \). Due to the multiplier estimate, the reaction force \( \lambda J \) is always an element of orthogonal complement \( O^*S(x) \).

The multiplier estimate cannot be derived from any principles, but its use is beneficial because the right-hand side of Equation (34) depends only on \( x \). Hence, the multiplier estimate gives a one-to-one mapping from \( x \) to \( \lambda \), and the multipliers can be eliminated from the problem. Additionally, the multipliers are carried over between the different analyses and follow the stress analysis stages smoothly. This occurs because, when the DR process is judged to converge, the reaction forces can be computed by Equation (34).

Due to the multiplier estimate, only the initial configurations of position and velocity (i.e., \( x \) and \( q \)) are given and updated in the geodesic DR process as well as in the existing DR method.

3.3. Discrete parallel transportation operator

As shown in Figure 8, the discrete parallel transportation operator \( \varphi(q) \) projects \( q \) to \( TS(x) \) and is given by

\[
\varphi(q) = \frac{|q|}{|(I_n - J^+J)q|} (I_n - J^+J)q. \tag{35}
\]

This operator is similar to the projected gradient, but preserves the norm of the given vector. This characteristic contributes to the energy conservation law in the geodesic DR. In order to achieve energy conservation, the kinetic energy must at least be conserved when no damping nor external forces are applied. If the norm of the velocity is not preserved during the projection, the kinetic
energy decreases, even if no damping is applied. This means that, if the same operator as the projected gradient is applied to the velocity, unexpected damping effects might arise.

Similarly to the projected gradient, the discrete parallel transportation operator guarantees that the velocity always points in the direction that does not change the values of \( g_1(x), \ldots, g_r(x) \).

In the geodesic DR, \( \varphi(q_t) \) should be used for computation of the entity \( \theta \) between the velocity and the acceleration, i.e.,

\[
\theta_t = \frac{\varphi(q_t)^T r_t}{|\varphi(q_t)||r_t|} \tag{36}
\]

When the damping coefficient \( \gamma \) is given by a damping function \( \gamma(\theta) \), the second line of the geodesic DR (Equation (25)) is replaced with

\[
q_{t+1} = \gamma(\theta_t) \varphi(q_t) + \beta r_t. \tag{37}
\]

### 3.4. Pull back operator

Figure 9 shows that pull back operator, \( \psi(x) \), tries to pull \( x \) back onto \( \bar{S} \). Unlike the projected gradient and the discrete parallel transportation operators, \( \psi(x) \) is not a simple operator. Instead, it can be defined by the iterative algorithm given in flowchart 2 shown in Figure 10.

In the flowchart, \( b \) represents the residual of constraint conditions defined by

\[
b(x) = \begin{bmatrix}
g_1(x) - \bar{g}_1 \\
\vdots \\
g_r(x) - \bar{g}_r
\end{bmatrix}. \tag{38}
\]

Also \( \Delta x = -J^+ b \) is a minimum norm solution for a system of linear equations

\[
J(x) \Delta x = -b(x), \tag{39}
\]

which is a linear approximation of \( b(x + \Delta x) = 0 \). Because Equation (39) is a linear approximation, ideally, \( x \) should ideally be close to \( \bar{S} \) in each step. It should be noted that \( \Delta x = -J^+ b \) is an element of the orthogonal complement \( OS(x) \).

The flowchart in Figure 10 further shows that \( s \), a counter for iteration cycles, and \( |b| \), the norm of residual of constraint conditions, are updated in every step and used to evaluate whether the cycles are continued or not. The threshold values for both the counter, \( s \), and the norm, \( |b| \), denoted by \( \bar{N} \) and \( \bar{\xi} \), respectively, in the flowchart, are given. The iteration cycles are terminated when the iteration count \( s \) reaches \( \bar{N} \) or when \( |b| \) becomes less than \( \bar{\xi} \).

Even if the projected gradient and discrete parallel transportation operators make the acceleration \( r_t \) and velocity \( q_t \), always point toward the direction that does not change the values of \( g_1(x), \ldots, g_r(x) \), small errors in constraint conditions accumulate because an isomaniold \( S(x) \) is generally not flat. Hence, they are not sufficient to keep the values of \( g_1(x), \ldots, g_r(x) \) constant. This phenomenon means that, even if \( x \) truly belongs to the constraint isomaniold \( S \) at step \( t \),
\( \psi(x) \): Pull back operator projects \( x \) onto \( \bar{S} \) by iteratively solving \( b(x + \Delta x) = 0 \).

Figure 9.

![Flowchart 2: Pull back operator.](image)

Figure 10.

\( x \) gradually detaches itself from \( \bar{S} \) in later steps, unless at least one iteration of the pull back is performed in each step. However, if \( S(x) \) is sufficiently close to \( \bar{S} \), i.e., \( |b| \) is sufficiently small, only single iteration is sufficient for the pull back because Equation (39) is an appropriate approximation of \( b(x + \Delta x) = 0 \). Therefore, the basic number of iterations in the pull back is one. More than one iterations is called when \( |b| \) is greater than \( \xi \).

Having defined the three operators (projected gradient, discrete parallel transportation and pull back) in the sections 3.2-3.4, the geodesic DR can be summarized. In each step, the external force \( \omega \) and velocity \( q \) are projected to tangent spaces, \( T^*S(x) \) and \( TS(x) \). The norm of \( q \) is preserved before and after the projection, while a simple orthographic projection is applied to external force \( \omega \). By using the projected vectors, a single iteration, which is exactly the same operation as the existing DR, is performed. After this operation, if the new \( x \) is sufficiently close to the constraint isomaniold \( \bar{S} \), the single step pull back operation is performed. If not, the iterative pull back operation is performed until the iteration cycle count reaches a prescribed maximum number or...
the norm of constraint condition residual becomes less than a prescribed threshold. This approach is further illustrated in flowchart 3 (Figure 11).

Eventually, the conditions \( \phi(\omega) = 0 \) and \( S(x) = \hat{S} \) are satisfied with an acceptable tolerance and the geodesic DR method converges. Due to the definition of the projected gradient, when the projected gradient vanishes, a relation of \( \omega = \omega J^+J \) is established. Choosing the multipliers as \( \lambda = -\omega J^+ \), we have \( \omega = -\lambda J \) and, hence, obtain \( \omega + \lambda J = 0 \), which is the first stationary condition. This condition can be expressed as \( \tau(x, \lambda) = 0 \). It is clear that \( S(x) = \hat{S} \) ensures the second stationary condition. As shown in Figure 12, both \( \omega \) and \( \lambda J \) belong to \( O^*S(x) \). On the other hand, \( \phi(\omega) \), which belongs to \( T^*S(x) \), disappears. Figure 12 also indicates that the external force \( \omega \) and the reaction force \( \lambda J \) cancel each other out. Therefore, their decomposition disappears, which is a reinterpretation of \( \phi(\omega) = 0 \) in terms of statics.

In contrast with the existing DR method, the geodesic DR technique evaluates \( \omega(x) \) and \( J(x) \). However, \( J(x) \) contains gradients of the constrained quantities, and the geodesic DR only assesses the first derivatives of the functions. Therefore, neither the existing nor the presented geodesic DR methods require the computation of the second derivatives of the function. As a tradeoff between avoiding computation or estimation of second derivatives, the DR method is might be considered slow when compared with the Newton methods family (e.g. Newton-Raphson and quasi-Newton methods). Nevertheless, the extension presented in this paper has potentially large impact as it is independent of the computation of second derivatives. The presented extension improves the performance of the existing DR method, which has been widely adopted by academia and industry for the form-finding and nonlinear analysis of pre-stressed structural systems.

4. DISCUSSION OF THE GEODESIC DYNAMIC RELAXATION METHOD

In parallel with section 2, the major characteristics (conservation of energy, damping coefficient and geodesic generation) of the geodesic DR method are discussed.
Figure 12: Equilibrium established in a problem of equilibrium with equality constraint conditions: External force $\omega$ and reaction force $\lambda J$ cancel each other out at $x$.

4.1. Conservation of Energy

First, the effect of applying the discrete parallel transportation operator to the velocity in the geodesic DR method on the energy conservation law is investigated. We apply the geodesic DR method to the following numerical example:

$$f(x, y, z) = 0.1(z + 5) \rightarrow \min,$$

s.t. $$\left( x - \frac{R_A}{\sqrt{x^2 + z^2}} \right)^2 + y^2 + \left( z - \frac{R_A}{\sqrt{x^2 + z^2}} \right)^2 = R_B^2,$$

where the function $f$ represents a gravity potential, and the constraint condition is an implicit representation of a torus with $R_A$ as the major and $R_B$ as the minor radii. The initial configurations are

$$x_0 = \begin{bmatrix} R_A + R_B \sin \nu & R_B \cos \nu & 0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

where $\nu$ is a parameter to move $x_0$ along a meridian. It should be noted that $x_0$ satisfies the constraint condition and, hence, $S(x) = \bar{S}$ is satisfied in the initial step.

We use the following values: $R_A = 3$, $R_B = 2$, and $\nu = 2.34$. For the time step, we set $\beta = 0.1$, and $N = 10$ and $\xi = 0.01$ for the pull back. The trajectory generated by the geodesic DR method with no damping ($\gamma = 1.0$) is shown in Figure 13 (a). This figure shows that the trajectory does not converge nor diverge. The history of the total and kinetic energies versus the number of iterations in the geodesic DR method is plotted in Figure 13 (b). This figure illustrates that the total energy of the system is roughly conserved throughout the computation. If the projected gradient is applied to the velocity, instead of the discrete parallel transportation, the total energy gradually decreases because the projected gradient always decreases the norm of the velocity without amplifying it.

4.2. Damping coefficient

The effect of adopting different damping functions (viscous, kinetic and drift) on the convergence of the geodesic DR method is investigated in the case study presented in section 4.1. The trajectories and histories of the energies versus number of time steps are plotted in Figure 14. These plots illustrate the effects of the three damping approaches well. When kinetic damping is solely applied, the kinetic energy is not absorbed until the kinetic energy reaches a local maximum. On the other hand, when viscous damping is solely applied, a small amount of the kinetic energy is continuously dissipated along the process. The drift damping approach exhibits the beneficial characteristics of both the kinetic and the viscous damping methods. Throughout the geodesic DR process, it continuously dissipates kinetic energy (cfr., viscous damping). However, the rate of dissipation of kinetic energy is adjusted dynamically, and it rapidly increases once the process hits a local peak in kinetic energy (cfr., kinetic damping).
4.3. Geodesic generation

In the geodesic DR method, the law of inertia on a curved space can be reproduced by setting the external force, $\omega$, to 0 and damping coefficient, $\gamma$, to 1.0. In other words, we can draw a geodesic on an implicitly represented surface (or on a higher dimensional manifold) by using the geodesic DR.
method. Because both the external force and damping are not given, the geodesic DR method can be simplified for this specific purpose, as illustrated in flowchart 4 (Figure 15).

Using the algorithm presented in Figure 15, geodesics are generated on the torus defined in section 4.1. The same initial configuration as Equation (41) is used but the value of $\nu$ is varied from 0 to $2\pi$. The other parameters are kept constant (i.e. for the time step, $\beta = 0.1$, the pull-back, $N = 10$ and $\xi = 0.01$, the radii of torus, $R_A = 3$, and $R_B = 2$).

In Figure 16, the top row figures show the trajectories on the torus generated by geodesic DR approach while the bottom row figures show geodesics on the torus given by a program developed with commercially available software, Mathematica® [42]. The program algorithm solves an ordinary differential equation that gives geodesics on a parametric surface, given by an explicit representation.

Even though our algorithm is technically very different from the Mathematica® program algorithm, both routines visually give the same results with the exception of one part indicated by ellipses in the figure. Hence, we claim that our method can generate a geodesics on an implicit surface or, more broadly, that the presented geodesic DR method generates geodesics when no force is applied.

The discussion of the characteristics of the geodesic DR method suggests a number of similarities and differences with the existing DR method. Both methods preserve the total energy of the system when no damping is applied and work with different damping approaches (viscous, kinetic and drift damping). However, unlike the DR method, the geodesic DR produces geodesics as opposed to generating straight lines. From a differential geometry perspective, geodesics are one of the natural generalizations of straight lines to curved spaces. Hence, the geodesic DR can be thought as a natural extension of the existing DR method.

5. CASE STUDY

To show the validity and accuracy of the geodesic DR method, we carried out the form finding of an existing, pre-stressed membrane structure, Tanzbrunnen (1957, Cologne, Germany, Frei Otto). The original design of Tanzbrunnen is a minimal surface structure based upon the geometry of a physical soap film model. This system consists of a radially, pre-stressed, ridge-valley membrane with an oculus inner ring. The membrane is supported by six masts, equally spaced along the outer tension cable ring of the membrane. The ridge-valley configuration is achieved by 12 pre-stressed cables, radially positioned in the membrane. These cables run from the inner oculus tension ring either (i) over the tops of the masts or (ii) to a point, positioned on the membrane’s outer tension ring between two mast heads. The masts’ footings themselves are pinned to the foundation and the mast heads are further stabilized with stay cables. The points on the membrane outer tension
Figure 16. Geodesics on a torus (a): Geodesics computed by our method for different values of $v$. (b): Curves generated by a Mathematica® program [42]. The arrows show velocity at the initial point. Except one part indicated by the ellipses, they visually match well.

ring are also stabilized by one stay cable going to the foundation. The membrane and the cables on the inner and outer tension rings are pre-stressed. The length of the masts and the stay cables are constrained. The total lengths of the 12 pre-stressed cables, positioned in the membrane surface, are also constrained to a fixed length.
5.1. Form finding process

The initial configuration, used for \( x_0 \) in the form finding process, is given in Figure 17 (a). Figure 17 (b) shows which nodes are fixed, which elements are pre-stressed and which elements have been assigned constraint conditions. \( L \) and \( S \), shown in Figure 17 (b), represent the length between two nodes of a linear element, and the area of a triangular element defined by three nodes, respectively. Additionally, \( w \) is a weight coefficient that is assigned and used to make the elements larger or smaller. In this case study, we only consider constraints related to length.

The energy function to be minimized by the geodesic DR method is expressed as follows:

\[
f(x) = w_s \sum_{j \in D} S_j^s + w_1 \sum_{j \in E} L_j^1 + w_2 \sum_{j \in F} L_j^1,
\]

where \( w_s \), \( w_1 \), and \( w_2 \) are weight coefficients for membrane and the outer and inner tension rings, respectively, and \( s \) and \( t \) are exponents of \( S \) and \( L \), respectively. Typically, these exponents have values of 1 or 2 in form finding problems. The choice of these exponents is discussed further in this section. Additionally, \( D \), \( E \) and \( F \) are the sets of (triangular or linear) elements in the membrane, outer and inner ring, respectively.

In the model, there are 504 free nodes and 18 nodes (i.e. 6 mast bottoms and 12 connections of the stay cables to the foundation) pinned in all three directions. Therefore, the degree of freedom, i.e. the dimension of \( x \), in this problem is 1512. As shown by Figure 17 (c), for the pre-stressed elements (membrane and tension ring cables) the weight coefficients \( w_s \), \( w_1 \), and \( w_2 \) are treated as design parameters to control and study the shape of the membrane structure. On the other hand, for the elements for which the lengths are constrained (masts, stay cables and radial cables), the lengths themselves are treated as design parameters.

Two different types of constraint conditions are incorporated into the design problem. The first type is a simple length constraint condition of a linear element and the second type is a total length constraint condition of a set of linear elements. There are 18 constraint conditions of the first type and 12 constraint conditions of the second. Hence, there are 30 multipliers packed together in \( \lambda \).

In order to attain radial symmetry in the structure, the constraint conditions are assembled to one group per 6 conditions, and hence there are 5 groups of constraint conditions. In each group, a common constraint value is used for six constraint conditions, such as \( \bar{L}_1, \ldots, \bar{L}_5 \).

For example, the first group contains the six masts with conditions described as

\[
L_{145}(x) = \bar{L}_1 \\
\vdots \\
L_{150}(x) = \bar{L}_1
\]

Similarly, the second and third groups contain constraint conditions of the same type for stay cables, in which \( \bar{L}_2 \) and \( \bar{L}_3 \) are used for constraint values. The fourth and fifth groups contain six conditions for six pre-stressed cables, in which \( \bar{L}_4 \) and \( \bar{L}_5 \) are used to prescribe total lengths of cables, such as

\[
\sum_{j=163}^{168} L_j(x) = \bar{L}_4 \\
\vdots \\
\sum_{j=193}^{198} L_j(x) = \bar{L}_4
\]

We carried out the geodesic DR method with specific design parameters: \( \{ w_s = 0.8, w_1 = 5, w_2 = 12 \} \) and \( \{ L_1 = 4.69, L_2 = 1.764, L_3 = 5.176, L_4 = 5.357, L_5 = 5.092 \} \) with exponents \( \{ s = 1, t = 1 \} \). We used \( \beta = 0.05 \) for the time step and \( N = 50 \) and \( \xi = 0.0001 \) for pull back. However, an unexpected result was obtained, as shown by Figure 17 (d). This result can be attributed
to the exponents chosen. We tested another choice of exponents \( \{ s = 1, t = 2 \} \) while holding the other parameters to the same values. Although improvement was observed, very small and distorted triangular elements have been noted in the converged solution model, as shown by Figure 17 (e). This occurrence is due to the singularity problem, which has been reported in the context of minimal surface problems [43]. When the Newton method family is used to solve minimal surface problems, this singularity prevents the process from converging because the stiffness matrix becomes harder to invert as the method iterates. On the contrary, the DR method can directly minimize the sum of element areas without confronting this singularity problem. However, as one can observe in Figure 17 (d) and (e), this singularity still causes unexpected results even if the DR method is employed for minimization. Therefore, regardless of the DR method’s ability to solve the problem, we should not ignore the singularity problem and adopt a strategy to avoid it.

Wüchner and Bletzinger tackled this singularity problem [43] in the minimal surface problem. Their approach was to modify the minimal surface problem carefully to make the modified problem as similar to the original as possible, but the singularity no longer exists in the modified problem. Because further discussion of minimal surface problem is outside the scope of this paper, instead, we modified the original problem largely and simply chose \( \{ s = 2, t = 2 \} \) as exponents. This choice is equivalent to giving surface stress density [44] to the triangular elements and force density to the linear elements [45].

Thus, a stable shape for Tanzbrunnen structure was obtained, as shown by Figure 17 (f) with specific design parameters: \( \{ s = 2, t = 2 \}, \{ w_s = 0.8, w_1 = 1, w_2 = 12 \} \) and \( \{ L_1 = 4.69, L_2 = 1.764, L_3 = 5.176, L_4 = 5.357, L_5 = 5.092 \} \). Note that we changed \( w_1 \) as well as the exponents to obtain appropriate proportions for the structure. For form-finding problems, especially for problems with equality constraint conditions, a higher exponent for \( s \) and \( t \) sometimes provides finer result. For further discussion, the interested reader is referred to [46]. With higher exponents, the physical meaning of energy function becomes rather unclear. However, by following a systematic static approach of statics, the converged stress state under static equilibrium is further clarified, as described in section 5.2.

With the specific design parameters used for Figure 17 (f), we tested different damping approaches. When no damping is applied (i.e. \( \gamma = 1.0 \)), conservation of energy is observed, as shown in Figure. 18. When using viscous (\( \gamma = 0.98 \)), kinetic, or drift damping, the geodesic DR method converged to a stable equilibrium state after 5000 iterations. This took 3 minutes and 30 seconds with a Core i5 2.56 GHz. Further plots of history of energies and norm of projected gradient, \( \varphi(\omega) \), are given in Figure 19. The plots in Figure 19 clearly show that the shapes of history curves of the geodesic DR process, with drift damping, are smooth and approximate those of viscous damping. However, the convergence rate is more comparable to that of kinetic damping. Additional plots about pull back operator in the geodesic DR process are given in Figure 20. Figure 20 further shows that the residual of the constraint conditions reduces to a sufficiently small amount shortly after the geodesic DR process starts. Except for the initial phases of the convergence process, only one single iteration is needed for pull back in the later phases.

5.2 Stress analysis

When the geodesic DR process converges, a stable equilibrium state is achieved. The stress state in this particular equilibrium condition can be further analyzed. The obtained stress state only implies a ratio of forces in a constructed (pre-tensioned) structure, but it does not disclose the absolute magnitude of those forces. Moreover, the ratios between the forces in the constructed (pre-tensioned) structure might be different from those in the form-found system, where fictitious material properties are used. In general, the pre-stress in a real tensile surface realistically depends on a number of parameters such as the stiffnesses of the chosen technical textile, the method of pre-stressing the membrane (which can include more than one pre-stressing device), and the way the different structural components are connected. Nevertheless, we demonstrate the stress analysis for the problem discussed in section 5.1 because of its didactic value. The force ratios and stress field obtained by the following stress analysis would be, at least, consulted in the later engineering stages.
Figure 17. Numerical model of Tanzbrunnen (F. Otto, 1956, Kologne, Germany): (a) an initial configuration – dimensions are expressed in [m], (b) strain energies and constraint conditions given to the elements, (c) design parameters to control the size of the elements, (d) a numerical result with a specific combination of design parameters.
The variation of Lagrangian composed of the problem takes the form

$$\delta L = \sum_{j \in D} \sigma_j \delta S_j + \sum_{j \in E,F,G} n_j \delta L_j = 0,$$

(45)

Figure 18. Energy conservation law observed in the form finding process ($\beta = 0.001$, $\gamma = 1.0$, $N = 50$, $\xi = 0.0001$).

Figure 19. Plots of the form finding process using geodesic DR method with (a) viscous ($\gamma = 0.98$), (b) kinetic, and (c) drift damping. Top: Plots of energies versus number of iterations. The shape of the total energy curve in (c) is smooth and close to (a), but its convergence rate is rather closer to (b). Bottom: Plots of the norm of projected gradient $\phi (\omega)$ versus number of iterations (0 is optimum). Drift damping exhibits equal efficiency to kinetic damping.
Figure 20. Plots about pull back operator in the same analysis as Figure 19. Top: number of iterations performed in pull back operator in each step of geodesic DR process with (a) viscous ($\gamma=0.98$), (b) kinetic, and (c) drift damping. Except for the first step, only one single iteration is required in most of later steps. Bottom: Norm of residual of constraint conditions (0 is optimum). With any of three damping approaches, the satisfaction of constraint conditions is guaranteed shortly after computation starts.

where $\delta$ is a variational operator with respect to $x$, and $G$ is a set of linear elements in the constraint conditions. This means that the variations of the terms with respect to constraint conditions, which have the form of either $\lambda_k \delta L_k$ or $\lambda_k \sum_{j \in D} \delta L_j$, is merged with the second term because they have the same form as the second term. In terms of static mechanics, Equation (45) is called the principle of virtual work. In Equation (45), it can be observed that $\sigma_j$ represents the magnitude of the surface stress acting on the $j$-th triangular element, and $n_j$, the magnitude of the axial force acting on the $j$-th linear element.

Comparing Equation (45) with the variation of $w_s \sum_{j \in D} S_j^2$ that was used in the form-finding analysis, we obtain

$$\sigma_j = 2w_s S_j. \quad (46)$$

This expression means that the surface stress is proportional to element area and $w_s = \frac{1}{2} \frac{\sigma_j}{S_j}$, which is a half of the surface stress density defined in [44]. Similarly, comparing Equation (45) with the variations of $w_1 \sum_{j \in E} L_j^2$ and $w_2 \sum_{j \in F} L_j^2$, we obtain

$$n_j = 2w_1 L_j \text{ for } j \in E, \quad \text{and } n_j = 2w_2 L_j \text{ for } j \in F, \quad (47)$$

which are the axial forces acting in the linear elements on the outer and inner rings, respectively. Therefore, the axial force is proportional to element length and $w = \frac{1}{2} \frac{n_j}{L_j}$, which is half of the force density defined in [45]. For the linear elements in the constraint conditions, in which either each length of the linear element or total length of a set of linear elements is constrained to a prescribed value, we obtain

$$n_j = \lambda_1, \cdots, \text{ or } n_j = \lambda_5 \quad (48)$$
which also represent axial forces. As such, the multipliers generally represent the magnitudes of reaction forces produced by structural components which are modeled using equality constraint conditions. The multipliers have a negative sign for the compression mast and a positive sign for the tension elements. This sign convention is not applied in any stage of the form-finding process and should be checked after the geodesic DR process has converged. On the contrary, when values greater than 0 are given to the weight coefficients, Equations (46) and (47) are guaranteed to have the appropriate sign, which indicates that the elements are in a tension state.

The stress tensor field in the elements can be further analyzed as follows. We write \{A_{ij}\} for the matrix whose elements are denoted by \(A_{ij}\). A matrix with lower indices, such as \{A_{ij}\}, is used to represent the covariant components of a 2nd-order tensor. On the other hand, a matrix with upper indices, such as \{B^{ij}\}, is used to represent contra-variant components of a tensor. Additionally, we employ alternative notations with Einstein summation convention [47,48], in which summation symbols are omitted because double indices usually imply summation.

First, we assume that a natural coordinate system \((\theta^1)\) or \((\theta^1, \theta^2)\) [49] is adequately set up on a linear or triangular element and denote the domain of \((\theta^1)\) or \((\theta^1, \theta^2)\) on each element \(\Omega\). Then, for each linear or triangular element, we define the metric tensors (or first fundamental forms) \(g_{ij}\) by

\[
g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},
\]

where \(g_{ij} = g_i \cdot g_j\) and \(g_i\) represent natural (covariant) basis on the elements. For each linear or triangular element, the length and the area of the elements are defined by

\[
L = \int_{\Omega} dv^1, \quad S = \int_{\Omega} dv^2,
\]

where \(dv^1\) and \(dv^2\) are defined in a common form as

\[
dv^N = \sqrt{g} d\theta^1 \cdots d\theta^N
\]

and \(g\) is the determinant of \(\{g_{ij}\}\).

In addition, we define inverse matrices of \(\{g_{ij}\}\),

\[
\{g^{ij}\} = \left[ \begin{array} {cc} \frac{1}{g_{11}} \\ g_{21} & g_{22} \end{array} \right],
\]

or \(\{g^{ij}\} = \frac{1}{g} \left[ \begin{array} {cc} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{array} \right].
\]

In order to calculate the variations of \(L\) and \(S\) systematically, we use the relation \(\delta dv^N = \frac{1}{2} \{g^{\alpha \beta}\} : \{\delta g_{\alpha \beta}\} dv^N\), where we define the inner product between two matrices by \(\{A^\alpha{}^\beta\} : \{B_{\alpha \beta}\} = \sum_{\alpha \beta} A^\alpha{}^\beta B_{\alpha \beta}\). It should be noted that the ranges of indices change according to the dimensions of the element, i.e. \(\alpha, \beta \in \{1\}\) for linear elements and \(\alpha, \beta \in \{1, 2\}\) for triangular elements. When the Einstein summation convention is used, this is simply written as \(\delta dv^N = \frac{1}{2} g^{\alpha \beta} \delta g_{\alpha \beta} dv^N\). As a result, we obtain

\[
\delta L = \frac{1}{2} \int_{\Omega} \{g^{\alpha \beta}\} : \{\delta g_{\alpha \beta}\} dv^1, \quad \delta S = \frac{1}{2} \int_{\Omega} \{g^{\alpha \beta}\} : \{\delta g_{\alpha \beta}\} dv^2.
\]

The virtual work done by Cauchy stress tensor takes the form

\[
\delta w = \frac{1}{2} \int_{\Omega} \{T^\alpha{}^\beta\} : \{\delta g_{\alpha \beta}\} dv^N,
\]

where \(T^\alpha{}^\beta\) is the contra-variant component of the Cauchy stress tensor denoted by \(T = \sum_{\alpha} \sum_{\beta} T^\alpha{}^\beta \mathbf{g}_\alpha \otimes \mathbf{g}_\beta\), where \(\otimes\) represents a dyadic (or tensor) product of the base vectors. When the Einstein summation convention is used, it is simply denoted by \(T = T^\alpha{}^\beta \mathbf{g}_\alpha \otimes \mathbf{g}_\beta\).

Comparing Equations (53) and (54) with the virtual works, that were denoted by \(\delta w_j = n_j \delta L_j\) or \(\delta w_j = \sigma_j \delta S_j\) ignoring the summation convention, we obtain

\[
T = n_j I \quad \text{or} \quad T = \sigma_j I.
\]
where \( I = \sum_\alpha \sum_\beta g^{\alpha\beta} g_\alpha \otimes g_\beta \) is a unit tensor defined on each element. When the Einstein summation convention is used, it is denoted by \( I = g^{\alpha\beta} g_\alpha \otimes g_\beta \).

If all the triangle elements take the same value for \( \sigma_j \) at the same time, the global membrane stress field, over all the triangle elements, can be described by \( T = \bar{\sigma} I \) with a unique common scalar \( \bar{\sigma} \). This representation is a uniformly distributed, isotropic stress field, which is preferred in the early stage of design process of membrane structures [50]. This special stress field is also known as the minimal surface stress field, observed in physical soap film experiments. The obtained membrane stress \( \sigma_j = 2w_s S_j \) indicates that elements that have different element areas take different values for \( \sigma_j \). However, due to the minimization process of \( \sum_{j \in \bar{D}} S_j^2 \), the deviation of element areas in the obtained result is very small, and, hence, the obtained shape is expected to be close to the one obtained by a physical minimal surface soap film models.

6. CONCLUSIONS AND AVENUES FOR FURTHER RESEARCH

In this paper, the geodesic DR method was formulated and discussed. This approach is a novel extension of the existing DR method because it allows us to incorporate equality constraint conditions in force-modelled systems. In both the existing and geodesic DR methods, the total energy is preserved when no damping is given.

Drift damping was also introduced as a new damping approach. Drift damping is essentially a combination of two existing typical damping techniques, i.e., viscous and kinetic damping. This approach represents a novel way to dynamically adjust the damping coefficient between acceleration and deceleration. The drift damping technique inherits the smoothness from the viscous damping and dynamic adjustment feature from the kinetic damping. Hence, the drift damping is as robust as viscous damping and as efficient as kinetic damping. In both existing and geodesic DR methods, viscous, kinetic and drift damping can be used.

The existing DR generates straight lines by setting the external force to zero and damping coefficient to one (i.e., no damping is given). In contrast, the geodesic DR generates geodesics, which are a natural extension of straight lines to curved spaces. As an interesting byproduct, the geodesic DR method can be used to generate geodesics on implicitly represented surfaces.

The validity of the proposed, novel, geodesic DR method is demonstrated with a benchmark case study of a pre-stressed, ridge-valley system supported by six masts.

Although the equality constraint conditions in the case study were limited to length constraints of the masts, stay cables and membrane radial cables, the geodesic DR method could be enhanced with other types of equality constraint conditions. For example, when total volume of air (in the case of a pneumatic system) or the angle of hinge type joint are constrained, the Lagrange multipliers can generate the air pressure acting on the membrane or moment acting on the hinge, respectively.

Finally, because energy conservation was observed in the geodesic DR method, we would like to suggest that the geodesic DR method could be used to solve dynamic problems, in which case the mass matrix should not be ignored in its formulation.

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References


A. A BENCHMARK CASE WITHOUT CONSTRAINT CONDITIONS

Although this paper mainly described the extension of the DR method to problems of equilibrium with constraint conditions, we provided a flexible framework for damping, which is also beneficial for problems without constraint conditions. However, other than for kinetic damping, the DR method changes its performance according to the adjustable parameters in the viscous and drift damping formulations (equations (13) and (15) respectively). Those adjustable parameters are (i) the constant value for $\gamma$ in the viscous damping and (ii) the $\gamma - \theta$ relation in the drift damping.

In this section, by using a simple case study, we further clarify the performance of the DR method with different parameter choices in each damping approach. Figure 21 (a) shows the initial configuration of the benchmark model, which lies in the x-y plane. The model has 25x2x625 nodes (including the 4 corner nodes that are pinned in all directions) and 24x25x2=1200 linear elements. The energy function to be minimized is,

$$ f(x) = \sum_j L_j^2(x) \rightarrow \min. $$

This case study has a unique solution which is shown by Figure 21 (b). As $f(x)$ is a quadratic function of $x$, the same problem can be solved by a single computation of a linear system of equations. This linear method is known as the Force Density Method (FDM) [45]. Using this method the same geometry shown in Figure 21 (b) is obtained. The result by FDM is thought to be sufficiently precise and the absolute value of $f(x)$ at the solution by FDM is 6130.0579.

For this case study, Figure 22 (a) shows the history curves of the total and kinetic energy and the norm of gradient of $f$ in a DR process with kinetic damping. We used $\beta = 0.1$ for the time step and the same number was used for the other damping approaches. The absolute value of $f(x)$ at the step 2000 is 6130.058. Because the kinetic damping has no parameters, there is only a unique trial. Since there is no need to adjust parameters, this damping approach, with solely kinetic damping, might be preferred to the other approaches. However, the other damping approaches, especially the drift damping, have the potential to be superior to the kinetic damping in terms of convergence efficiency due to its adjustable parameter.

Figure 22 (b) and (c) show plots of the time step versus the total and kinetic energy and the norm of gradient of the unconstrained case study for 4 variations for both the viscous and the drift damping approaches. The absolute value of $f(x)$ at the step 2000 is 6130.072, for Figure 22 (b-1) and (b-2), and 6130.058 for (c-3), (a-1), (a-2) and (a-3). Figure 22 (b-4) and (c-4) are not judged as converged at time step 2000. In addition, FDM solved the problem in 280ms while 8 seconds of computation was needed for 2000 step iterations of DR. Hence, FDM is much faster than DR.
method. However, FDM can only solve types of quadratic problems like Equation (56) while DR can solve more general nonlinear problems. Furthermore, while FDM creates $621 \times 621$ matrix, DR only creates 1242×1 vectors. Hence, DR is not as memory consuming as FDM. Comparing and contrasting the results of these 9 analyses (1 kinetic, 4 viscous and 4 drift damping), yield the following observations:

- Although the analysis using the kinetic damping approach demonstrates superior performance in terms of reducing the norm of gradient, the peak of kinetic energy is 4 to 6 times larger compared to the peaks in drift damping (c-1) and (c-2), which exhibit close convergence efficiency to the kinetic damping.
- As shown in Figure 22 (b-2), an approach that adopts viscous damping, with a specific parameter giving the best performance in the first stage of the minimization, performs poorly in the later steps. This observation suggests that the dynamic adjustment included in the kinetic and drift damping approach is desirable in the DR process.
- Although the drift damping requires an adjustment of $\gamma - \theta$ relation curve, it is possible to achieve a performance close to the one obtained by kinetic damping. However, employing this close drift damping approach yields kinetic energy peaks that are considerably smaller than the peaks obtained using kinetic damping.
Figure 22: Results of a benchmark test: (a) History curves with kinetic damping. (b) History curves with viscous damping with different parameters. (c) History curves with drift damping with different $\gamma - \theta$ relations.